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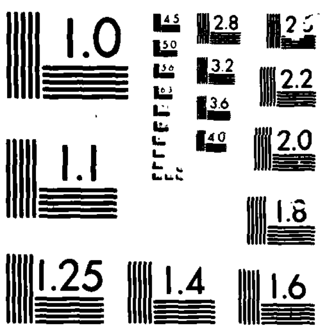

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THE POLYNOMIALS IN THE LINEAR SPAN OF INTEGER TRANSLATES OF A COMPACTLY SUPPORTED FUNCTION

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ABSTRACT

Algebraic facts about the space of polynomials contained in the span of integer translates of a compactly supported function are derived and then used in a discussion of the various quasi-interpolants from that span.

AMS (MOS) Subject Classifications: 41A15, 41A63, 41A25

Key Words: Box splines, multivariate, splines, quasi-interpolant, semi-discrete convolution

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SIGNIFICANCE AND EXPLANATION

The linear span of integer translates of a fixed compactly supported function φ provides a particularly simple model *of an approximating family of the finite element type.* The approximating power of such a span (or, more precisely, of its scaled versions) has been known for some time to be characterizable in terms of the space π_φ of polynomials it contains.

Recent work on box splines has provided concrete examples of interest in a multivariate setting and so rekindled interest in the space π_φ. The report derives and extends specific information about π_φ contained in recent work by Dahmen & Micchelli, and by Chui, Diamond, Jetter, Lai and Ward, but does so without reference to specific properties (such as piecewise polynomiality, or factorizability of the Fourier transform) of φ.

Understanding, in the simplest possible and most efficient terms, of the approximation power of such spaces may provide the necessary insight into approximation by smooth piecewise polynomials on regular, and perhaps even not so regular, partitions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE POLYNOMIALS IN THE LINEAR SPAN OF INTEGER TRANSLATES OF A COMPACTLY SUPPORTED FUNCTION

Carl de Boor

This note was stimulated by the recent papers [CD85], [CJW85], and [CL85] in which the authors take a new look at the space of integer translates of box splines and, in particular, introduce and highlight the **commutator** of a locally supported pp function φ of several variables. The intent of this note is to offer alternative proofs of some of these results, and to point to some connections with earlier work (e.g., [BH82/3], [DM83], [BJ84]), but also to focus more attention on the space π_φ of **all** polynomials contained in the span of the integer translates of the box spline (or other compactly supported) φ.

The first section collects simple algebraic facts about π_φ and the action of the linear map

$$\varphi *' : f \mapsto \sum_{j \in \mathbb{Z}^d} \varphi(\cdot - j) f(j)$$

on it.

The second section records that π_φ is invariant under differentiation and translation, and brings yet another characterization of π_φ, this time in terms of the Fourier transform of φ.

The final section makes use of these facts about π_φ in a discussion of the various quasi-interpolants available.

Throughout, I will use standard multi-index notation. I find it convenient to use the special symbol $[\![\,]\!]^\alpha$ for the **normalized monomial of degree** α, i.e., for the map given by the rule

$$[\![\,]\!]^\alpha : \mathbb{R}^d \longrightarrow \mathbb{R} : x \mapsto x^\alpha / \alpha!.$$

With this,

$$\pi_\alpha := \operatorname{span}([\![\,]\!]^\beta)_{\beta \le \alpha}$$

denotes the space of all polynomials of degree $\le \alpha$, and

$$\pi_k := \operatorname{span}([\![\,]\!]^\beta)_{|\beta| \le k}, \quad \pi_{<k} := \operatorname{span}([\![\,]\!]^\beta)_{|\beta| < k}, \quad \pi := \operatorname{span}([\![\,]\!]^\beta)$$

have similarly obvious meaning.

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1. The polynomials Consider the span of integer translates of a compactly supported function φ on $\mathbb{R}^d$, i.e.,

$$S := S_\varphi := \{\varphi * c : c \in \mathbb{R}^{\mathbb{Z}^d}\}. \tag{1.1}$$

Here I use the convolution product notation

$$\varphi * c := \sum_{j \in \mathbb{Z}^d} \varphi(\cdot - j)c(j) \tag{1.2}$$

since there is no danger of confusion with either the continuous or the discrete convolution product. I find it convenient to use the special notation

$$\varphi *' f := \varphi * f_{|\mathbb{Z}^d} = \sum_{j \in \mathbb{Z}^d} \varphi(\cdot - j)f(j) \tag{1.3}$$

in case f is a function on $\mathbb{R}^d$, in order to stress the semi-discrete character of this product. Further, since the restriction to $\mathbb{Z}^d$ of a function on $\mathbb{R}^d$ occurs often here, I will employ the abbreviation

$$f_| := f_{|\mathbb{Z}^d}$$

for it.

The asymmetry in the semi-discrete convolution product (1.3) is not all that strong since, after all,

$$\varphi *' f = f *' \varphi \quad \text{on } \mathbb{Z}^d.$$

This implies, e.g., that, for $f \in \pi$ (hence $f *' \varphi \in \pi$),

$$\varphi *' f = f *' \varphi \iff \varphi *' f \in \pi,$$

hence

$$\pi_\varphi := \{f \in \pi : \varphi *' f \in \pi\} = \{f \in \pi : \varphi *' f = f *' \varphi\}. \tag{1.4}$$

It also implies that

$$\varphi *' f = f *' \varphi \quad \text{for all } f \in S, \tag{1.5}$$

since, for $f = \varphi * c$,

$$\begin{aligned} \varphi *' f &= \varphi * (\varphi_| * c) \\ &= \varphi * (c * \varphi_|) \\ &= (\varphi * c) * \varphi_| = f *' \varphi. \end{aligned}$$

As a consequence, one gets the inclusion

$$\pi \cap S \subseteq \{f \in \pi : \varphi *' f = f *' \varphi\} = \pi_\varphi, \tag{1.6}$$

and the conclusion that

$$\varphi *' : f \mapsto \varphi *' f$$

maps π_φ into $\pi \cap S$. This implies that there must be equality throughout (1.6) as soon as the linear map

$$L := \varphi *'_{|\pi_\varphi}$$

can be shown to be 1-1. But that is easy to do under the assumption that φ is **normalized**, i.e.,

$$\sum_{j \in \mathbb{Z}^d} \varphi(j) = 1.$$

For, under this assumption,

$$\begin{aligned} \text{for } f \in \pi_\varphi, \quad \varphi *' f &= f \textstyle\sum_{j\in\mathbb{Z}^d} \varphi(j) - \sum_{j\in\mathbb{Z}^d} \big(f - f(\cdot - j)\big)\varphi(j) \\ &\in f + \pi_{<\deg f} \end{aligned} \tag{1.7}$$

since, for each j, $f - f(\cdot - j) \in \pi_{<\deg f}$.

The salient facts of this discussion are gathered in the following.

Proposition 1. *If φ is normalized, then*

$$\begin{aligned} \pi_\varphi : &= \{f \in \pi : \varphi *' f \in \pi\} = \{f \in \pi : \varphi *' f = f *' \varphi\} \\ &= \pi \cap S = \{f \in \pi : \varphi *' f \in f + \pi_{<\deg f}\}. \end{aligned} \tag{1.8}$$

Further, $L := \varphi *'_{|\pi_\varphi}$ is onto, and

$$U := 1 - L \tag{1.9}$$

is degree-reducing. In particular,

$$L(\pi_\varphi \cap \pi_\alpha) = \pi_\varphi \cap \pi_\alpha. \tag{1.10}$$

As a consequence, $U^k = 0$ on

$$\pi_{\varphi,k} := \pi_\varphi \cap \pi_{<k}.$$

Therefore

$$\left(L_{|\pi_{\varphi,k}}\right)^{-1} = (1 + U + \cdots + U^{k-1})_{|\pi_{\varphi,k}}. \tag{1.11}$$

Note that π_φ **is necessarily finite dimensional**, since φ is compactly supported. Precisely, for any bounded set D, the set

$$A(D) := \{\alpha \in \mathbb{Z}^d : \varphi(\cdot - \alpha)_{|D} \neq 0\}$$

is finite, hence if D also has interior, then

$$\dim \pi_\varphi = \dim \pi_{\varphi|D} \leq \# A(D) < \infty.$$

The sharpest bound attainable this way for a piecewise continuous φ would be

$$\dim \pi_\varphi \leq \max_x \# A(\{x\}). \tag{1.11}$$

In any case, this implies that

$$L^{-1} = 1 + U + U^2 + \cdots ,$$

with the Neumann series actually finite.

The assumption that φ be normalized is no real restriction except when

$$\sum_{j \in \mathbb{Z}^d} \varphi(j) = 0.$$

In this case, (1.7) shows L to be degree-reducing, hence in particular, not invertible. Consequently, $\pi \cap S$ may be strictly smaller than π_φ. For example, with $\varphi = 1$ on $[-1,0[$, $= -1$ on $[0,1[$, and $= 0$ otherwise, $\pi_\varphi = \pi_0 \neq \{0\} = \pi \cap S$.

2. Invariance Denote by E the **multivariate shift**, i.e.,

$$E^\alpha f := f(\cdot - \alpha), \quad \alpha \in \mathbb{Z}^d.$$

While it is obvious that $\varphi *'$ commutes with E, hence π_φ is invariant under E, some of the other properties of π_φ derivable from this fact may not be as immediate.

Proposition 2.1 *The linear map $L = \varphi *'_{|\pi_\varphi}$ commutes with differentiation, hence with translation, i.e.,*

$$LD^\alpha = D^\alpha L, \ \forall \alpha \in \mathbb{Z}_+{}^d, \quad E^y L = LE^y, \ \forall y \in \mathbb{R}^d. \tag{2.1}$$

Proof Since π_φ is a finite-dimensional polynomial subspace, there exists, for each $\alpha \in \mathbb{Z}_+{}^d$, a weight sequence w of finite support so that

$$D^\alpha = \sum_{\beta \in \mathbb{Z}_+{}^d} w(\beta) E^\beta \ \text{ on } \ \pi_\varphi. \tag{2.2}$$

(E.g., with ℓ_i the Lagrange polynomials for the points $0, \ldots, k :=$ max deg π_φ, we have

$$p = \sum_{0 \le \beta(j) \le k} \ell^\beta E^\beta p(0)$$

for all $p \in \pi_k(\mathbb{R}) \otimes \cdots \otimes \pi_k(\mathbb{R}) \supseteq \pi_\varphi$, hence $w(\beta) := D^\alpha \ell^\beta(0)$, all β, would do.) Thus, $LE = EL$ implies $LD = DL$. But this finishes the proof since

$$E^y = \sum_\alpha [\![y]\!]^\alpha D^\alpha. \tag{2.3}$$

Remark The argument shows that any E-invariant polynomial subspace is D-invariant, hence even translation-invariant, i.e., for any linear subspace P of π,

$$\begin{aligned} \forall \alpha \in \mathbb{Z}^d \ E^\alpha P \subseteq P \quad &\Longrightarrow \quad \forall \alpha \in \mathbb{Z}^d \ D^\alpha P \subseteq P \\ &\Longrightarrow \quad \forall y \in \mathbb{R}^d \ E^y P \subseteq P \end{aligned}. \tag{2.4}$$

Corollary *π_φ is D-invariant and translation-invariant.*

As a simple consequence, consider the polynomials g_α defined in [CJW85] by the recurrence

$$g_\alpha(x) := x^\alpha - \sum_{j \in \mathbb{Z}^d} \varphi(j) \sum_{\beta \ne \alpha} \binom{\alpha}{\beta} (-j)^{\alpha - \beta} g_\beta(x) \tag{2.5}$$

and then shown to satisfy

$$x^\alpha = \sum_{j \in \mathbb{Z}^d} g_\alpha(j)\varphi(x-j) \tag{2.6}$$

in case $|\alpha| < m$ and $\pi_{<m} \subset \pi_\varphi$. In other words, $g_\alpha = L^{-1}()^\alpha$. Therefore, on first reading, I thought that the recurrence relation (2.5) was a consequence of the fact that L is "unit upper triangular". In fact, the recurrence can be derived from the identity $DL = LD$.

For this, recall that the **Appell sequence** for a continuous linear functional μ on $C(\mathbb{R}^d)$ with $\mu(1) = 1$ is, by definition, the sequence (g_α) determined by the conditions

$$g_\alpha \in \pi_\alpha, \quad \mu D^\beta g_\alpha = \delta_{\beta\alpha}.$$

There is, in fact, exactly one such sequence for given μ since the linear system

$$\mu D^\beta \Big(\sum_{\gamma \le \alpha} [\![\,]\!]^\gamma a_\gamma \Big) = \delta_{\beta\alpha}$$

for the power coefficients (a_γ) for g_α has a unit triangular coefficient matrix. Backsusbstitution therefore provides the formula

$$g_\alpha = [\![\,]\!]^\alpha - \sum_{\beta \ne \alpha} \mu [\![\,]\!]^{\alpha-\beta} g_\beta, \tag{2.5'}$$

whose correctness can also be verified directly by induction on α:

$$\begin{aligned} \mu D^\gamma g_\alpha &= \mu D^\gamma [\![\,]\!]^\alpha - \textstyle\sum_{\beta\ne\alpha} \mu [\![\,]\!]^{\alpha-\beta} \mu D^\gamma g_\beta \\ &= \mu [\![\,]\!]^{\alpha-\gamma} - \mu [\![\,]\!]^{\alpha-\gamma} = 0 \end{aligned}$$

for $\gamma < \alpha$, while $\mu D^\alpha g_\alpha = \mu D^\alpha [\![\,]\!]^\alpha = \mu(1) = 1$. With existence and uniqueness established, facts about the Appell sequence, such as symmetries which reflect those of μ, or that $D^\beta g_\alpha = g_{\alpha-\beta}$, follow immediately.

In our case, $\mu : f \mapsto \varphi *' f(0)$, hence, for $[\![\,]\!]^\alpha \in \pi_\varphi$,

$$\delta_{\beta\alpha} = \mu D^\beta g_\alpha = \varphi *' (D^\beta g_\alpha)(0) = D^\beta(\varphi *' g_\alpha)(0),$$

which, together with the fact that $\varphi *' g_\alpha \in L\pi_\alpha = \pi_\alpha$, shows that

$$\varphi *' g_\alpha = [\![\,]\!]^\alpha. \tag{2.6'}$$

The resulting different normalization of g_α as compared with (2.5) avoids all those factorials.

Dahmen and Micchelli [DM83] consider the polynomial space

$$\{p \in \pi : p(D)\hat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \backslash 0\}. \tag{2.7}$$

with $\hat{\varphi}$ the Fourier transform of φ. It seems slightly more convenient to consider instead

$$\Pi_\varphi := \{p \in \pi : p(-iD)\hat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d\backslash 0\}.$$

They prove that any affinely invariant (i.e., translation- and scale-invariant) subspace of (2.7), hence of Π_φ, is contained in π_φ. But their proof can be made to show more.

Proposition 2.2 *π_φ is the largest E-invariant subspace of Π_φ.*

Proof The proof in [DM83] is based on the observation that, by Poisson's summation formula,

$$\varphi *' p(x) = \sum_\alpha \varphi(x-\alpha)p(\alpha) =: \sum_\alpha \psi(\alpha) = \sum_\alpha \hat{\psi}(2\pi\alpha),$$

while, for any $p \in \pi$, the function $\psi : y \mapsto \varphi(x-y)p(y)$ has the Fourier transform

$$\hat{\psi}(y) = e^{-ixy}\Big(p(x-iD)\hat{\varphi}\Big)(-y).$$

If now $p \in P$, with P an E-invariant (hence D-invariant) subspace of Π_φ, then

$$\hat{\psi}(2\pi\alpha) = \Big(p(x-iD)\hat{\varphi}\Big)(2\pi\alpha) = \sum_\beta [\![x]\!]^\beta \Big(D^\beta p(-iD)\hat{\varphi}\Big)(2\pi\alpha) = 0$$

for $\alpha \neq 0$, hence

$$\begin{aligned}\varphi *' p(x) &= \Big(p(x-iD)\hat{\varphi}\Big)(0)\\ &= \sum_\alpha D^\alpha p(x)[\![-iD]\!]^\alpha \hat{\varphi}(0)\\ &= p(x)\hat{\varphi}(0) + \sum_{|\alpha|>0} D^\alpha p(x)[\![-iD]\!]^\alpha \hat{\varphi}(0).\end{aligned} \tag{2.8}$$

showing that $\varphi *' p \in \pi$, i.e., $p \in \pi_\varphi$.

On the other hand, if $p \in \pi_\varphi$, then

$$\varphi *' p = \sum_\alpha e^{-2\pi i\alpha()}\Big(p(\cdot - iD)\Big)\hat{\varphi}(-2\pi\alpha)$$

is a polynomial, and this is possible only if

$$p(\cdot - iD)\hat{\varphi}(2\pi\alpha) = 0 \quad \forall\alpha \neq 0.$$

showing that $p \in \Pi_\varphi$.

Corollary $\pi_{<k} \subset \pi_\varphi$ *and* $\pi_{<h} \subset \pi_\psi \implies \pi_{<k+h} \subset \pi_{\varphi*\psi}$.

Proof If $|\alpha+\beta| < k+h$, then either $|\alpha| < k$ or else $|\beta| < h$, hence $|\gamma| < k+h$ implies

$$[\![-iD]\!]^{\gamma}(\widehat{\varphi * \psi})(2\pi j) = \sum_{\alpha+\beta=\gamma} [\![-iD]\!]^{\alpha}\hat{\varphi}(2\pi j) [\![-iD]\!]^{\beta}\hat{\psi}(2\pi j) = 0$$

for $j \in \mathbb{Z}^d \backslash 0$.

While π_φ has been shown in [BH82/3] to be scale-invariant in case φ is a box spline, it is not clear that π_φ is necessarily scale-invariant for arbitrary φ. For this, I note that a polynomial subspace P is scale-invariant if and only if P **stratifies**, i.e., $P = \sum_k P \cap \pi_k^0$, with

$$\pi_k^0 := \text{ span } \left([\![\,]\!]^{\alpha}\right)_{|\alpha|=k}.$$

Hence, $\text{span}\{[\![\,]\!]^{2,0} + [\![\,]\!]^{0,1}, [\![\,]\!]^{1,0}, 1\}$ provides a simple example of an E-invariant polynomial subspace which is not scale-invariant.

3. Quasi-interpolants The space π_φ is of interest because it characterizes the local approximation order obtainable from S, or, more precisely, from the **scale** (S_h) associated with S. To recall,

$$S_h := \sigma_h(S),$$

with

$$\sigma_h f : x \mapsto f(x/h).$$

Further, the **local approximation order** of S is the largest k for which

$$dist(f, S_h) = O(h^k)$$

for all smooth f, with the distance measured in some norm, e.g., the max-norm on some bounded domain,, and the support of the approximation to f within h of the support of f.

In [FS69], Fix and Strang give a characterization of the local approximation order from the scale (S_h) which, in the terms of Section 1, can be phrased thus: it is the largest k for which

$$U := 1 - \varphi *' \quad \text{is degree-reducing on } \pi_{<k}. \tag{3.1}$$

Proposition 1 shows that we can state this condition more simply as

$$\pi_{<k} \subseteq \pi_\varphi. \tag{3.2}$$

To be precise, [FS69] consider the "controlled" approximation order, which turns out to be the same as the local approximation order; cf. [BJ84].

Fix and Strang use in their proof a quasi-interpolant whose construction relies on Fourier transform arguments which, in a univariate context, can already be found in Schoenberg's basic spline paper [S46] and which appear in the proof of Proposition 2.2. This makes it easy to recall their construction here.

Define the quasi-interpolant Q on π by the rule

$$Qf := \varphi *' Ff$$

with

$$Ff := \sum_\alpha a_\alpha (-iD)^\alpha f$$

and $a_\alpha := (D^\alpha(1/\hat\varphi))(0)$ the Taylor coefficients for $1/\hat\varphi$. Dahmen and Micchelli [DM83] prove that Q reproduces any affinely invariant subspace of (2.7), but, again, their argument supports a stronger claim, viz. that

$$Q|_{\pi_\varphi} = 1. \tag{3.3}$$

For, if $p \in \pi_\varphi$, then also $Fp \in \pi_\varphi$ since π_φ is D-invariant; hence, by (2.8),

$$\begin{aligned} Qp &= \sum_\alpha (D^\alpha F p)[\![-iD]\!]^\alpha \hat{\varphi}(0) \\ &= \sum_\alpha \sum_\beta a_\beta (-iD)^{\alpha+\beta} p \ [\![D]\!]^\alpha \hat{\varphi}(0) \\ &= \sum_\gamma (-iD)^\gamma p \sum_{\alpha+\beta=\gamma} [\![D]\!]^\beta (1/\hat{\varphi})(0) [\![D]\!]^\alpha \hat{\varphi}(0) \\ &= \sum_\gamma (-iD)^\gamma p \ \delta_{0\gamma} = p. \end{aligned}$$

The construction is finished by noting that (3.3) only depends on the action of F on π_φ, hence a local quasi-interpolant on smooth functions which reproduces π_φ can be obtained in the form

$$Qf := \pi_\varphi *' (\lambda * f), \tag{3.4}$$

with

$$(\lambda * f)(x) := \lambda f(\cdot + x), \tag{3.5}$$

and λ any locally supported linear functional which agrees on π_φ with $p \mapsto Fp(0)$.

The construction idea in [BH] seems more direct: There the locally supported bounded linear functional (on whatever normed linear space X you may wish to carry out approximation from $S \cap X$) is constructed as an extension of the linear functional

$$p \mapsto (L^{-1}p)(0). \tag{3.6}$$

Since $L = \varphi *'_{|\pi_\varphi}$ commutes with E, so does L^{-1}. Thus, for $p \in \pi_\varphi$,

$$(L^{-1}p)(j) \ = \ (L^{-1}p(\cdot + j))(0) = (\lambda * p)(j),$$

hence

$$Qp = \varphi *' (L^{-1}p) = p.$$

In order to obtain a quasi-interpolant of the optimal order k, the extension λ only needs to match (3.6) on $\pi_{<k}$. For example, one obtains the Strang-Fix quasi-interpolant by expressing the extension as a linear combination of the linear functionals

$$f \mapsto (-iD)^\alpha f(0), \quad |\alpha| < k, \tag{3.7}$$

i.e., in the form

$$\lambda f = \sum_{|\alpha|<k} a_\alpha (-iD)^\alpha f(0).$$

The weights a_α are uniquely determined by the requirement that this linear functional match (3.6) since (3.7) is maximally linearly independent over $\pi_{<k}$. In particular,

$$a_\alpha = L^{-1}[\![i \cdot]\!]^\alpha(0) = i^\alpha g_\alpha(0),$$

by (2.6'). This shows, incidentally, that

$$[\![D]\!]^\alpha(1/\hat{\varphi})(0) = i^\alpha g_\alpha(0).$$

If point evaluation is continuous on X, then the linear functional λ can be written as a linear combination of evaluations at integer points near 0. For, by (1.11),

$$L^{-1}{}_{|\pi_{<k}} = (1 + U + \cdots + U^{k-1})_{|\pi_{<k}},$$

while, from (1.9),

$$(Uf)(j) = (c * f_|)(j),$$

with

$$c := \delta - \varphi_|$$

and δ the unit sequence, i.e., $\delta(j) = \delta_j$. Hence

$$(L^{-1}p)(0) = p^{[k]}(0)$$

with $p^{[k]}$ obtained inductively in the following computation:

$$p^{[r]} := \begin{cases} 0, & \text{if } r = 0; \\ p_| + c * p^{[r-1]}, & \text{if } r > 0. \end{cases} \tag{3.8}$$

This gives

$$(L^{-1}p)(0) = \sum_{j \in \mathbb{Z}^d} C(j)p(j), \quad \text{all} \quad p \in \pi_{<k}$$

with the weight sequence C of finite support since c has finite support.

This construction was arrived at by different means by Chui and Diamond [CD85], who added the following very useful observation. **If φ is symmetric, then U reduces the degree by at least 2**, since (1.7) can then be written in the form

$$\text{for } f \in \pi_\varphi, \quad \varphi *' f = f \sum_{j \in \mathbb{Z}^d} \varphi(j) + \sum_{j \in \mathbb{Z}^d} (f(\cdot + j) - 2f + f(\cdot - j))\varphi(j)/2. \tag{1.7'}$$

This implies that, on $\pi_{\varphi,k}$, already $U^{\cdot k}$ vanishes, hence only half the iteration (3.8) is necessary in this case.

Even for a symmetric φ, the support of the resulting λ may be far from minimal. Since we are only interested in extending a linear functional from π_φ, a support consisting of (dim π_φ) points is sufficient. These points can be chosen from $\mathbb{Z}^d$ since $\mathbb{Z}^d$ is total for π. It would be interesting to find out whether they could be chosen as neighbors.

Such questions of minimal support for λ have been answered quite elegantly by Dahmen and Micchelli in case φ is a box spline. They find in [DM85] that the (dim π_φ) integer points in the (right-continuous) support of φ are linearly independent over π_φ, and so conclude the existence of an extension from π_φ involving just these (dim π_φ) point evaluations.

I note that the quasi-interpolant construction in [BJ84] takes the opposite tack. Instead of constructing an appropriate λ as a linear combination of certain point evaluations, a compactly supported function $\psi \in S$ is constructed there so that already $\psi *'$ reproduces π_φ.

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